A Proof of Wavelength Conversion Not Improving the Lagrangian Bound of the Static RWA Problem

James Yiming Zhang, Jing Wu, *Senior Member, IEEE,* Gregor v. Bochmann, *Fellow, IEEE,* and Michel Savoie

***Abstract*—The fact that wavelength conversion hardly improves the performance of static routing and wavelength assignment (RWA) in Wavelength Division Multiplexing (WDM) networks has been observed in many previous studies. However, other than simulation results, until now there was no formal proof of such fact. In this paper, we formally prove that wavelength conversion does not improve the Lagrangian bound of the static RWA problem.**

***Index Terms*—Wavelength division multiplexing, wavelength conversion, routing and wavelength assignment, performance bound**

I. Introduction

In wavelength-routed WDM networks, a lightpath is switched from an input fibre link to an output fibre link in the optical domain. If wavelength conversion is available at an intermediate WDM switch node, a lightpath may change its wavelength by using a wavelength converter. Otherwise, a lightpath must use the same wavelength.

The static RWA problem is to find out optimized RWA schemes for all (or a selected subset of) lightpath requests under certain design objectives. Once a lightpath uses a wavelength channel on a given fibre link, other lightpaths cannot use the same wavelength channel.

Previous studies revealed that wavelength conversion's contribution to the static RWA problem is minimal [1-12]. Computations have shown that most likely, the use of wavelength conversion can be avoided by re-arranging RWA schemes since all lightpath requests are known in advance. However, until now there was no formal proof of such fact.

In this paper, we prove that wavelength conversion does not improve the Lagrangian bound of the static RWA problem. This bound is obtained by solving the Dual Problem (DP), which is generated by relaxing the original RWA problem (i.e., the primal problem) using the Lagrangian relaxation method [13]. Since previous studies showed that such a bound can be very close to the objective function value of the static RWA problem [3], we can conclude that the difference of the achieved objective function values between the cases with and without wavelength conversion is very marginal.

Manuscript received October 7, 2008.

James y. Zhang and Gregor v. Bochmann are with the School of Information Technology and Engineering, University of Ottawa, Ottawa, Ontario, Canada, K1N 6N5, Email: {yizhang, bochmann}@site.uottawa.ca; Jing Wu and Michel Savoie are with the Communications Research Centre Canada, Ottawa, Ontario, Canada, K2H 8S2, Email:{jing.wu, michel.savoie}@crc.gc.ca.

II. Lagrangian Bound of the Static RWA Problem

A. A Formulation of the Static RWA Problem with and without Wavelength Conversion

We formulate the static RWA problem as a penalty-minimization problem. Two classes of the static RWA problem are investigated. In the first class, wavelength conversion is available at all nodes, while in the second class, no wavelength conversion is available at any node. Similar formulations are used for the two problems, except that in the second problem, the number of wavelength converters at each node is limited to zero.

Our network model consists of nodes interconnected by fibres in an arbitrary mesh topology. Each fibre has W wavelength channels. The fibre between nodes i and j is denoted by e_{ij}. The c^{th} wavelength channel on e_{ij} is denoted by w_{ijc} $(0 < c \leq W)$. The set E represents all fibre links in the network. Each link has a pair of fibres, one for each direction. The set V represents all the WDM switch nodes in the network. s_{sdn} denotes the n^{th} lightpath request between node pair (s, d). The set S represents all lightpath requests over the network.

We use the same penalty-based objective function (primal function) as [14], which efficiently formulates the design objectives. Our objective function is thus $\min\limits_{\mathrm{A},\Delta,\Phi}(J)$, where

$$J = \sum_{s_{sdn}\in S} \left[(1-\alpha_{sdn})\,P_{sdn} + \alpha_{sdn}C_{sdn}\right] \tag{1}$$

For each request s_{sdn}, either the penalty of rejecting it (P_{sdn}), or the penalty of using resources (C_{sdn}) to set up a lightpath is added to the objective function (J), depending on s_{sdn}'s admission status α_{sdn}. α_{sdn}=0, if s_{sdn} is rejected; and α_{sdn}=1, if s_{sdn} is admitted.

In addition to the design variables A=(α_{sdn}), we introduce the design variables Δ=(δ_{ijc}^{sdn}), where δ_{ijc}^{sdn} represents the use of w_{ijc} by s_{sdn}. If w_{ijc} is used by s_{sdn}, δ_{ijc}^{sdn} equals one; otherwise, δ_{ijc}^{sdn} equals zero. We use Δ_{sdn} for the wavelength assignment of s_{sdn}, i.e., $(\delta_{ijc}^{sdn})_{sdn}$. The total cost of using wavelength channels and wavelength converters is the cost of s_{sdn}, denoted by C_{sdn}:

$$C_{sdn} = \sum_{e_{ij}\in E}\sum_{0<c\leq W} d_{ij}\delta_{ijc}^{sdn} + \sum_{i\in V} o_i\phi_i^{sdn}, \quad \forall s_{sdn}\in S \tag{2}$$

where d_{ij} is the cost of using w_{ijc}, o_i is the cost of using a wavelength converter at node i, ϕ_i^{sdn} is a 0-1 integer variable, representing the use of a wavelength converter at node i by s_{sdn}. If a wavelength converter is used by s_{sdn}, ϕ_i^{sdn} equals one; otherwise, ϕ_i^{sdn} equals zero. We use Φ_{sdn} to denote the assignment of the wavelength converters to s_{sdn} (i.e., $\left(\phi_i^{sdn}\right)_{sdn}$) and use Φ to denote (Φ_{sdn}).

A solution to the static RWA problem $\min\limits_{\mathrm{A},\Delta,\Phi}(J)$ is denoted by $U = (\mathrm{A}, \Delta, \Phi)$. Its optimal solutions are denoted by $U^* = (\mathrm{A}^*, \Delta^*, \Phi^*)$.The above problem must conform to the following constraints,

a) Lightpath continuity constraints:
If a request is admitted, the lightpath assigned to it has to be continuous along a path between the source-destination pair.

$$\sum_{j\in V}\sum_{0<c\le W}\delta_{ijc}^{sdn}-\sum_{j\in V}\sum_{0<c\le W}\delta_{jic}^{sdn}=\begin{cases}\alpha_{sdn} & \text{if } i=s\\ -\alpha_{sdn} & \text{if } i=d\\ 0 & \text{otherwise}\end{cases},$$
$$\forall s_{sdn}\in S \tag{3}$$

b) Wavelength channel exclusive usage constraints:

$$\sum_{s_{sdn}\in S}\delta_{ijc}^{sdn}\le 1,\quad \forall e_{ij}\in E, 0<c\le W \tag{4}$$

c) Wavelength converter quantity constraints:

$$\sum_{s_{sdn}\in S}\phi_i^{sdn}\le F_i\quad \forall i\in V \tag{5}$$

If wavelength converters are installed at a node, they are configured in a shared-per-node manner [15], where any input or output port may use a wavelength converter if one is available. This configuration allows the best sharing of wavelength converters. If wavelength converters do not contribute to the optimization objective in this installation, it can be concluded that they do not contribute to the same optimization objective in any other installation. The number of wavelength converters at node i is denoted by F_i. Note that as long as $F_i \ge W \times d_i$, where d_i denotes the nodal degree of node i, the number of converters can accommodate any incoming traffic. The number of possibly used converters at a node must be no more than the number of installed converters at the node.

d) Wavelength conversion constraints:

$$\phi_j^{sdn}=\begin{cases}1 & \text{if } \exists m,k\in V \text{ and } b\ne a,\ \delta_{mja}^{sdn}=\delta_{jkb}^{sdn}=1\\ 0 & \text{otherwise}\end{cases},$$
$$\forall j\in V, j\ne s, j\ne d \tag{6}$$

A wavelength converter at an intermediate node j is used only when different wavelengths are assigned to s_{sdn} for the incoming and outgoing signals at this node.

B. A Bound of the Static RWA Problem Derived from its Lagrangian Dual Problem

In this section, we will present a method to derive a bound of the static RWA problem described in the previous section. We use the Lagrangian relaxation method to derive a DP for the static RWA problem.

We use the Lagrangian relaxation framework to derive a Lagrangian DP from the static RWA problem $\min\limits_{U}(J)$, by relaxing selected constraints. Lagrange multipliers ξ_{ijc} and λ_i are introduced in association with the constraints that represent resource limitations, that is, the wavelength channel exclusive usage constraints (4), and wavelength converter quantity constraints (5), respectively. The vectors of Lagrange multipliers (ξ_{ijc}) and (λ_i) are denoted as ξ and λ, respectively. The Lagrangian function L is defined as [13]:

$$L(U,\xi,\lambda)=J(U)+\sum_{e_{ij}\in E}\sum_{0<c\le W}\xi_{ijc}\left(\sum_{s_{sdn}\in S}\delta_{ijc}^{sdn}-1\right)$$
$$+\sum_{i\in V}\lambda_i\left(\sum_{s_{sdn}\in S}\phi_i^{sdn}-F_i\right), \tag{7}$$

where $\xi, \lambda \ge 0$. Note that L is a function of the Lagrange multipliers $m = (\xi, \lambda)$ and the design variables U.

The dual function $q(m)$ is defined as the infimum of L.

$$q\ (m)=\min_U\left(L\left(m,U\right)\right) \tag{8}$$

The Lagrangian DP is defined as $\max\limits_{m\ge 0}(q)$, subject to the constraints (3) and (6). We use q^* to denote the Lagrangian DP's optimal value. The corresponding optimal Lagrange multiplier values are denoted by $m^* = (\xi^*, \lambda^*)$. The optimal value of the Lagrangian DP is a lower bound of the primal problem [13]:

$$q^*\ (m^*)=\min_U\left(L\left(U,m^*\right)\right)\le\min_U\left(J\left(U\right)\right) \tag{9}$$

Corresponding to the two classes of static RWA problems, we derive two Lagrangian DPs, where in DP$_1$, the number of wavelength converters at each node is abundant; while in DP$_2$, the number of wavelength converters at each node must be restricted to zero.

Both Lagrangian DPs can be decomposed to the sub-problems, where each sub-problem corresponds to the RWA problem of one lightpath requests. The readers are referred to [14] for the solution details.

$$q\left(m\right)=\sum_{s_{sdn}\in S}\min_{\alpha_{sdn}}\{P_{sdn}\left(1-\alpha_{sdn}\right)+\alpha_{sdn}\min_{\Delta_{sdn},\Phi_{sdn}}$$
$$\sum_{e_{ij}\in E}\sum_{0<c\le W}\left[\delta_{ijc}^{sdn}\left(\xi_{ijc}+d_{ijc}\right)+\sum_{i\in V}\phi_i^{sdn}\left(\lambda_i+o_i\right)\right]\}$$
$$-\sum_{e_{ij}\in E}\sum_{0<c\le W}\xi_{ijc}-\sum_{i\in V}\lambda_iF_i \tag{10}$$

subject to the constraints in (3) and (6). Note that in DP$_2$, $\phi_i^{sdn}=0$, $\forall s_{sdn}\in S$, $\forall i\in V$.

III. Wavelength Conversion's Impact on the Bound of the Static RWA Problem

We now prove that the bounds obtained from the previous two Lagrangian DPs are the same, $(q_{DP_1})^* = (q_{DP_2})^*$, where $(q_{DP_1})^*$and $(q_{DP_2})^*$ denote the optimal values of DP$_1$ and DP$_2$, respectively. We consider a benchmark case where $o_i = 0$ and $F_i \ge W \times d_i$. From (10) and the definition of Lagrange multipliers [13], any $(q_{DP_1})^*$ for the cases other than the benchmark case must not be less than $(q_{DP_1})^*$ of the benchmark case. We will prove that even in the benchmark case, wavelength conversion does not improve the Lagrangian bound of the static RWA problem. Thus, the same conclusion applies to other wavelength converter configurations and cost parameters. We start with four Lemmas.

Lemma 1: In any of the optimal solutions (m^*, U^*) to DP_1, $(\lambda_i)^* = 0$, $\forall i \in V$.

Proof: Based on the definition of Lagrange multipliers [13], we have $(\lambda_i)^* \left(\sum_{s_{sdn} \in S} \left(\phi_i^{sdn}\right)^* - F_i \right) = 0, \forall i \in V$. Because we assume the number of wavelength converters at any node is abundant, the term $\sum_{s_{sdn} \in S} \left(\phi_i^{sdn}\right)^* - F_i$ must be strictly less than zero, $\forall i \in V$. Thus, $(\lambda_i)^* = 0$, $\forall i \in V$. ∎

Lemma 2: In any of the optimal solutions (m^*, U^*) to DP_1, $(\xi_{ij1})^* = (\xi_{ij2})^* = ... = (\xi_{ijW})^*$, $\forall e_{ij} \in E$.

Proof: if $0 \le (\xi_{ijl})^* < (\xi_{ijk})^*$, $\forall e_{ij} \in E, \forall k \ne l$, $0 < k \le W$, $0 < l \le W$, then $\left(\delta_{ijl}^{sdn}\right)^* = 1$ and $\left(\delta_{ijk}^{sdn}\right)^* = 0$. This is because from Lemma 1 and the zero cost assumption of wavelength converters, we have $(\lambda_i)^* + o_i = 0$, $\forall i \in V$. From (10), s_{sdn}'s least cost paths $(\Delta_{sdn})^*$ only include the wavelength channel with the least cost on a given link $e_{ij} \in E$. Thus, s_{sdn} chooses w_{ijl} (i.e. , $\delta_{ijl}^{sdn} = 1$), instead of w_{ijk}. Since $\left(\delta_{ijk}^{sdn}\right)^* = 0$, one of the optimal solutions satisfies $(\xi_{ijk})^* = 0$, $\forall k \ne l$, $0 < k \le W$, $0 < l \le W$, which contradicts the assumption $0 \le (\xi_{ijl})^* < (\xi_{ijk})^*$. ∎

Lemma 2 means that the Lagrange multipliers for the wavelength channels on a given link must be the same in an optimal solution to DP_1. Otherwise, all lightpaths through the link only use the least cost wavelength channels.

Lemma 3: At least one of the optimal solutions to DP_1 satisfies $\Phi^* = 0$.

Proof: If $\left(\phi_j^{sdn}\right)^* = 1$, because there exist $m, k \in V$ and $b \ne a$ that satisfy $\left(\delta_{mja}^{sdn}\right)^* = \left(\delta_{jkb}^{sdn}\right)^* = 1$, then by letting s_{sdn} use w_{jka}, instead of w_{jkb}, we get an optimal solution that satisfies $\left(\phi_j^{sdn}\right)^* = 0$. Lemma 2 ensures that this change satisfies all the constraints but does not influence the optimal value q^*. So the obtained solution is also one of the optimal solutions to DP_1. ∎

Lemma 3 means that even though wavelength conversion is available at no cost, at least one of the optimal solutions to DP_1 does not use the conversion.

Lemma 4: $q_{DP_1} \ge q_{DP_2}$.

Proof: In (10), because $\sum_{i \in V} \phi_i^{sdn} (\lambda_i + o_i) \ge 0$, we have $q_{DP_1} \ge q_{DP_2}$ for the same ξ. ∎

Lemma 4 means that regardless of the value of Lagrange multipliers λ, for the same value of the Lagrange multipliers ξ, the value of DP_1 must be no less than the value of DP_2.

With the above lemmas, we now prove our final theorems:

Theorem 1: An optimal solution $\left((m_{DP_1})^*, (U_{DP_1})^*\right)$ to DP_1 where $(\Phi_{DP_1})^* = 0$ is an optimal solution to DP_2.

Proof: If a DP_2's optimal solution $\left((m_{DP_2})^*, (U_{DP_2})^*\right)$, where $(\lambda_{DP_2})^* = 0$ and $(\Phi_{DP_2})^* = 0$, satisfies $q_{DP_2}\left((m_{DP_2})^*, (U_{DP_2})^*\right) < q_{DP_2}\left((m_{DP_1})^*, (U_{DP_1})^*\right)$, then for the left side, because $(\lambda_{DP_2})^* = 0$ and $(\Phi_{DP_2})^* = 0$, we get $q_{DP_2}\left((m_{DP_2})^*, (U_{DP_2})^*\right) = q_{DP_1}\left((m_{DP_2})^*, (U_{DP_2})^*\right)$; for the right side, using Lemma 4, we get $q_{DP_2}\left((m_{DP_1})^*, (U_{DP_1})^*\right) \le (q_{DP_1})^*$. Thus, we get $q_{DP_1}\left((m_{DP_2})^*, (U_{DP_2})^*\right) < (q_{DP_1})^*$, which means $\left((m_{DP_2})^*, (U_{DP_2})^*\right)$ produces a better dual value of DP_1 than $\left((m_{DP_1})^*, (U_{DP_1})^*\right)$ does. This contradicts the assumption that $\left((m_{DP_1})^*, (U_{DP_1})^*\right)$ is an optimal solution to DP_1. ∎

Theorem 1 means that if an optimal solution to DP_1 does not use wavelength conversion, it is an optimal solution to DP_2.

Theorem 2: $(q_{DP_1})^* = (q_{DP_2})^*$.

Proof: From Lemmas 1 and 3, we get $\sum_{i \in V} \left(\phi_i^{sdn}\right)^* \left((\lambda_i)^* + o_i\right) = \sum_{i \in V} (\lambda_i)^* F_i = 0$ in (10). Thus, $(q_{DP_1})^* = (q_{DP_2})^*$. ∎

IV. Conclusions

We proved that wavelength conversion does not improve the Lagrangian bound of the static RWA problem. Although it is not a direct proof that wavelength conversion does not improve the quality of the solutions to the static RWA problem, it implies that in solving the static RWA problem the contribution of wavelength conversion is very marginal, since the bound is very close to the achieved objective function value in most cases. Our results should apply to dynamic RWA problems that allow free rearrangement or removal of existing lightpaths.

References

[1] A. Zapata-Beghelli and P. Bayvel, "Dynamic versus static wavelength-routed optical networks" *J. Lightw. Technol.*, vol. 26, no. 20, pp. 3403-3415, 2008.

[2] B. Jaumard, C. Meyer, and X. Yu, "How much wavelength conversion allows a reduction in the blocking rate?" *J. Opt. Netw.*, vol. 5, no. 12, pp. 881-900. 2006.

[3] Y. Zhang, O. Yang, and H. Liu, "A Lagrangean relaxation and subgradient framework for the routing and wavelength assignment problem in WDM networks," *IEEE J-SAC*, vol. 22, no. 9, pp. 1752-1765, 2004.

[4] C. Assi, A. Shami, M. Ali, Z. Zhang, and X. Liu, "Impact of wavelength converters on the performance of optical networks," *Opt. Netw. Mag.*, vol. 3, no. 2, pp. 22-30, 2002.

[5] M.D. Swaminathan and K.N. Sivarajan, "Practical routing and wavelength assignment algorithms for all optical networks with limited wavelength conversion," *IEEE ICC 2002*.

[6] D. Schupke, "Off-line lightpath routing in WDM networks with different wavelength converter configurations," *IEEE HPSR 2002*.

[7] A. Thiagarajan and S. Somani, "An efficient algorithm for optimal wavelength converter placement on wavelength-routed networks with arbitrary topologies," *IEEE INFOCOM* 1999.

[8] E. Karasan and E. Ayanoglu, "Effects of wavelength routing and selection algorithms on wavelength conversion gain in WDM optical networks," *IEEE/ACM Trans. Netw.*, vol. 6, no. 2, pp. 186-196, 1998.

[9] S. Baroni and P. Bayvel, "Wavelength requirements in arbitrarily connected wavelength-routed optical networks," *J. Lightw. Technol.*, vol. 15, no. 2, pp. 242-251, 1997.

[10] N. Wauters and P. Demeester, "Design of the optical path layer in multiwavelength cross-connected networks," *IEEE J-SAC*, vol. 14, no. 5, pp. 881-892, 1996.

[11] N. Nagatsu, Y. Hamazumi, and K. Sato, "Optical path accommodation designs applicable to large scale networks," *IEICE Trans. Comm.*, vol. E78-B, no. 4, pp. 597-607, 1995.

[12] I. Chlamtac, A. Ganz, and G. Karmi, "Lightpath communications: An approach to high bandwidth optical WAN's," *IEEE Trans. Commun.*, vol. 40, no. 7, pp. 1171-1182, 1992.

[13] D.P. Bertsekas, *Nonlinear Programming* (2nd Ed.), Belmont, MA: Athena Scientific, 1999.

[14] J.Y. Zhang, J. Wu, G. Bochmann, and M. Savoie, "Grade-of-service differentiated static resource allocation schemes in WDM networks," *Opt. Swit. Netw.*, vol. 5, no. 2-3, pp. 107-122, 2008.

[15] G. Xiao and Y.W. Leung, "Algorithms for allocating wavelength converters in all-optical networks", *IEEE/ACM Trans. Netw.*, vol. 7, no. 4, pp. 545-557, 1999.